In this chapter we will consider an important class of problems in which the fluid is either at rest or moving in such a manner that there is no relative motion between adjacent particles. In both instances there will be no shearing stresses in the fluid, and the only forces that develop on the surfaces of the particles will be due to the pressure. Thus, our principal concern is to investigate pressure and its variation throughout a fluid and the effect of pressure on submerged surfaces. The absence of shearing stresses greatly simplifies the analysis and, as we will see, allows us to obtain relatively simple solutions to many important practical problems.

2.1 Pressure at a Point

As we briefly discussed in Chapter 1, the term pressure is used to indicate the normal force per unit area at a given point acting on a given plane within the fluid mass of interest. A question that immediately arises is how the pressure at a point varies with the orientation of the plane passing through the point. To answer this question, consider the free-body diagram, illustrated in Fig. 2.1, that was obtained by removing a small triangular wedge of fluid from some arbitrary location within a fluid mass. Since we are considering the situation in which there are no shearing stresses, the only external forces acting on the wedge are due to the pressure and the weight. For simplicity the forces in the $x$ direction are not shown, and the $z$ axis is taken as the vertical axis so the weight acts in the negative $z$ direction. Although we are primarily interested in fluids at rest, to make the analysis as general as possible, we will allow the fluid element to have accelerated motion. The assumption of zero shearing stresses will still be valid so long as the fluid element moves as a rigid body; that is, there is no relative motion between adjacent elements.
The equations of motion (Newton's second law, \( F = ma \)) in the \( y \) and \( z \) directions are:

\[
\sum F_y = p_y \delta x \delta z - p_x \delta x \delta s \sin \theta = \rho \frac{\delta x \delta y \delta z}{2} a_y
\]

\[
\sum F_z = p_z \delta x \delta y - p_x \delta x \delta s \cos \theta - \gamma \frac{\delta x \delta y \delta z}{2} = \rho \frac{\delta x \delta y \delta z}{2} a_z
\]

where \( p_x, p_y, \) and \( p_z \) are the average pressures on the faces, \( \gamma \) and \( \rho \) are the fluid specific weight and density, respectively, and \( a_y, a_z \) the accelerations. Note that a pressure must be multiplied by an appropriate area to obtain the force generated by the pressure. It follows from the geometry that

\[
\delta y = \delta z \cos \theta \quad \delta z = \delta s \sin \theta
\]

so that the equations of motion can be rewritten as

\[
p_y - p_x = \rho a_y \frac{\delta y}{2}
\]

\[
p_z - p_x = \left( \rho a_z + \gamma \right) \frac{\delta z}{2}
\]

Since we are really interested in what is happening at a point, we take the limit as \( \delta x, \delta y, \) and \( \delta z \) approach zero (while maintaining the angle \( \theta \)), and it follows that

\[
p_y = p_x = p_z = p_x
\]

or \( p_x = p_y = p_z \). The angle \( \theta \) was arbitrarily chosen so we can conclude that the pressure at a point in a fluid at rest, or in motion, is independent of direction as long as there are no shearing stresses present. This important result is known as Pascal's law named in honor of Blaise Pascal (1623–1662), a French mathematician who made important contributions in the field of hydrostatics. In Chapter 6 it will be shown that for moving fluids in which there is relative motion between particles (so that shearing stresses develop) the normal stress at a point, which corresponds to pressure in fluids at rest, is not necessarily the same in all directions. In such cases the pressure is defined as the average of any three mutually perpendicular normal stresses at the point.
Although we have answered the question of how the pressure at a point varies with direction, we are now faced with an equally important question—how does the pressure in a fluid in which there are no shearing stresses vary from point to point? To answer this question consider a small rectangular element of fluid removed from some arbitrary position within the mass of fluid of interest as illustrated in Fig. 2.2. There are two types of forces acting on this element: *surface forces* due to the pressure, and a *body force* equal to the weight of the element. Other possible types of body forces, such as those due to magnetic fields, will not be considered in this text.

If we let the pressure at the center of the element be designated as $p$, then the average pressure on the various faces can be expressed in terms of $p$ and its derivatives as shown in Fig. 2.2. We are actually using a Taylor series expansion of the pressure at the element center to approximate the pressures a short distance away and neglecting higher order terms that will vanish as we let $\delta x$, $\delta y$, and $\delta z$ approach zero. For simplicity the surface forces in the $x$ direction are not shown. The resultant surface force in the $y$ direction is

$$\delta F_y = \left( p + \frac{\partial p}{\partial y} \right) \delta x \delta z - \left( p + \frac{\partial p}{\partial y} \right) \delta x \delta z$$

or

$$\delta F_y = -\frac{\partial p}{\partial y} \delta x \delta y \delta z$$

Similarly, for the $x$ and $z$ directions the resultant surface forces are

$$\delta F_x = -\frac{\partial p}{\partial x} \delta x \delta y \delta z \quad \delta F_z = -\frac{\partial p}{\partial z} \delta x \delta y \delta z$$

---

**FIGURE 2.2**
Surface and body forces acting on small fluid element.
The resultant surface force acting on the element can be expressed in vector form as

\[ \delta F_s = \delta F_x \hat{i} + \delta F_y \hat{j} + \delta F_z \hat{k} \]

or

\[ \delta F_s = -\left( \frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} + \frac{\partial p}{\partial z} \hat{k} \right) \delta x \delta y \delta z \] (2.1)

where \( \hat{i}, \hat{j}, \) and \( \hat{k} \) are the unit vectors along the coordinate axes shown in Fig. 2.2. The group of terms in parentheses in Eq. 2.1 represents in vector form the pressure gradient and can be written as

\[ \frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} + \frac{\partial p}{\partial z} \hat{k} = \nabla p \]

where

\[ \nabla (\cdot) = \frac{\partial (\cdot)}{\partial x} \hat{i} + \frac{\partial (\cdot)}{\partial y} \hat{j} + \frac{\partial (\cdot)}{\partial z} \hat{k} \]

and the symbol \( \nabla \) is the gradient or "del" vector operator. Thus, the resultant surface force per unit volume can be expressed as

\[ \delta F_s \delta x \delta y \delta z = -\nabla p \]

Since the \( z \) axis is vertical, the weight of the element is

\[ -\delta W \hat{k} = -\gamma \delta x \delta y \delta z \hat{k} \]

where the negative sign indicates that the force due to the weight is downward (in the negative \( z \) direction). Newton's second law, applied to the fluid element, can be expressed as

\[ \sum \delta F = \delta m \quad a \]

where \( \Sigma \delta F \) represents the resultant force acting on the element, \( a \) is the acceleration of the element, and \( \delta m \) is the element mass, which can be written as \( \rho \delta x \delta y \delta z \). It follows that

\[ \sum \delta F = \delta F_z - \delta W \hat{k} = \delta m \quad a \]

or

\[ -\nabla p \delta x \delta y \delta z - \gamma \delta x \delta y \delta z \hat{k} = \rho \delta x \delta y \delta z \quad a \]

and, therefore,

\[ -\nabla p - \gamma \hat{k} = \rho \quad a \] (2.2)

Equation 2.2 is the general equation of motion for a fluid in which there are no shearing stresses. We will use this equation in Section 2.12 when we consider the pressure distribution in a moving fluid. For the present, however, we will restrict our attention to the special case of a fluid at rest.
2.3 Pressure Variation in a Fluid at Rest

For a fluid at rest \( a = 0 \) and Eq. 2.2 reduces to
\[
\nabla p + \gamma k = 0
\]
or in component form
\[
\frac{\partial p}{\partial x} = 0 \quad \frac{\partial p}{\partial y} = 0 \quad \frac{\partial p}{\partial z} = -\gamma \tag{2.3}
\]
These equations show that the pressure does not depend on \( x \) or \( y \). Thus, as we move from point to point in a horizontal plane (any plane parallel to the \( x-y \) plane), the pressure does not change. Since \( p \) depends only on \( z \), the last of Eqs. 2.3 can be written as the ordinary differential equation
\[
\frac{dp}{dz} = -\gamma \tag{2.4}
\]
Equation 2.4 is the fundamental equation for fluids at rest and can be used to determine how pressure changes with elevation. This equation indicates that the pressure gradient in the vertical direction is negative; that is, the pressure decreases as we move upward in a fluid at rest. There is no requirement that \( \gamma \) be a constant. Thus, it is valid for fluids with constant specific weight, such as liquids, as well as fluids whose specific weight may vary with elevation, such as air or other gases. However, to proceed with the integration of Eq. 2.4 it is necessary to stipulate how the specific weight varies with \( z \).

2.3.1 Incompressible Fluid

Since the specific weight is equal to the product of fluid density and acceleration of gravity \( (\gamma = \rho g) \), changes in \( \gamma \) are caused either by a change in \( \rho \) or \( g \). For most engineering applications the variation in \( g \) is negligible, so our main concern is with the possible variation in the fluid density. For liquids the variation in density is usually negligible, even over large vertical distances, so that the assumption of constant specific weight when dealing with liquids is a good one. For this instance, Eq. 2.4 can be directly integrated
\[
\int_{p_1}^{p_2} dp = -\gamma \int_{z_1}^{z_2} dz
\]
to yield
\[
p_2 - p_1 = -\gamma(z_2 - z_1)
\]
or
\[
p_1 - p_2 = \gamma(z_2 - z_1) \tag{2.5}
\]
where \( p_1 \) and \( p_2 \) are pressures at the vertical elevations \( z_1 \) and \( z_2 \), as is illustrated in Fig. 2.3. Equation 2.5 can be written in the compact form
\[
p_1 - p_2 = \gamma h \tag{2.6}
\]
or
\[
p_1 = \gamma h + p_2 \tag{2.7}
\]
where \( h \) is the distance, \( z_2 - z_1 \), which is the depth of fluid measured downward from the location of \( p_2 \). This type of pressure distribution is commonly called a *hydrostatic distribution*, and Eq. 2.7 shows that in an incompressible fluid at rest the pressure varies linearly with depth. The pressure must increase with depth to "hold up" the fluid above it.

It can also be observed from Eq. 2.6 that the pressure difference between two points can be specified by the distance \( h \) since

\[
\Delta p = p_1 - p_2 = \gamma h
\]

In this case \( h \) is called the *pressure head* and is interpreted as the height of a column of fluid of specific weight \( \gamma \) required to give a pressure difference \( p_1 - p_2 \). For example, a pressure difference of 10 psi can be specified in terms of pressure head as 23.1 ft of water (\( \gamma = 62.4 \) lb/ft\(^3\)), or 518 mm of Hg (\( \gamma = 133 \) kN/m\(^3\)).

When one works with liquids there is often a free surface, as is illustrated in Fig. 2.3, and it is convenient to use this surface as a reference plane. The reference pressure \( p_0 \) would correspond to the pressure acting on the free surface (which would frequently be atmospheric pressure), and thus if we let \( p_2 = p_0 \) in Eq. 2.7 it follows that the pressure \( p \) at any depth \( h \) below the free surface is given by the equation:

\[
p = \gamma h + p_0
\]

As is demonstrated by Eq. 2.7 or 2.8, the pressure in a homogeneous, incompressible fluid at rest depends on the depth of the fluid relative to some reference plane, and it is *not* influenced by the size or shape of the tank or container in which the fluid is held. Thus, in Fig. 2.4 the pressure is the same at all points along the line \( AB \) even though the container may have the very irregular shape shown in the figure. The actual value of the pressure along \( AB \) depends only on the depth, \( h \), the surface pressure, \( p_0 \), and the specific weight, \( \gamma \), of the liquid in the container.
Because of a leak in a buried gasoline storage tank, water has seeped in to the depth shown in Fig. E2.1. If the specific gravity of the gasoline is $SG = 0.68$, determine the pressure at the gasoline-water interface and at the bottom of the tank. Express the pressure in units of lb/ft$^2$, lb/in.$^2$, and as a pressure head in feet of water.

**EXAMPLE 2.1**

Because of a leak in a buried gasoline storage tank, water has seeped in to the depth shown in Fig. E2.1. If the specific gravity of the gasoline is $SG = 0.68$, determine the pressure at the gasoline-water interface and at the bottom of the tank. Express the pressure in units of lb/ft$^2$, lb/in.$^2$, and as a pressure head in feet of water.

**FIGURE E2.1**

**SOLUTION**

Since we are dealing with liquids at rest, the pressure distribution will be hydrostatic, and therefore the pressure variation can be found from the equation:

$$ p = \gamma h + p_0 $$

With $p_0$ corresponding to the pressure at the free surface of the gasoline, then the pressure at the interface is

$$ p_1 = SG \gamma_{H_2O} h + p_0 $$

$$ = (0.68)(62.4 \text{ lb/ft}^3)(17 \text{ ft}) + p_0 $$

$$ = 721 + p_0 \text{ (lb/ft}^2) $$

If we measure the pressure relative to atmospheric pressure (gage pressure), it follows that $p_0 = 0$, and therefore

$$ p_1 = 721 \text{ lb/ft}^2 \quad \text{(Ans)} $$

$$ \frac{p_1}{\gamma_{H_2O}} = \frac{721 \text{ lb/ft}^2}{62.4 \text{ lb/ft}^3} = 11.6 \text{ ft} \quad \text{(Ans)} $$

It is noted that a rectangular column of water 11.6 ft tall and 1 ft$^2$ in cross section weighs 721 lb. A similar column with a 1-in.$^2$ cross section weighs 5.01 lb.

We can now apply the same relationship to determine the pressure at the tank bottom; that is,

$$ p_2 = \gamma_{H_2O} h_{H_2O} + p_1 $$

$$ = (62.4 \text{ lb/ft}^3)(3 \text{ ft}) + 721 \text{ lb/ft}^2 $$

$$ = 908 \text{ lb/ft}^2 \quad \text{(Ans)} $$
Observe that if we wish to express these pressures in terms of absolute pressure, we would have to add the local atmospheric pressure (in appropriate units) to the previous results. A further discussion of gage and absolute pressure is given in Section 2.5.

The required equality of pressures at equal elevations throughout a system is important for the operation of hydraulic jacks, lifts, and presses, as well as hydraulic controls on aircraft and other types of heavy machinery. The fundamental idea behind such devices and systems is demonstrated in Fig. 2.5. A piston located at one end of a closed system filled with a liquid, such as oil, can be used to change the pressure throughout the system, and thus transmit an applied force \( F_1 \) to a second piston where the resulting force is \( F_2 \). Since the pressure \( p \) acting on the faces of both pistons is the same (the effect of elevation changes is usually negligible for this type of hydraulic device), it follows that \( F_2 = (A_2\/A_1)F_1 \). The piston area \( A_2 \) can be made much larger than \( A_1 \) and therefore a large mechanical advantage can be developed; that is, a small force applied at the smaller piston can be used to develop a large force at the larger piston. The applied force could be created manually through some type of mechanical device, such as a hydraulic jack, or through compressed air acting directly on the surface of the liquid, as is done in hydraulic lifts commonly found in service stations.

\[
\begin{align*}
  \frac{p_2}{\gamma_{\text{H}_2\text{O}}} &= \frac{908 \text{ lb/ft}^2}{62.4 \text{ lb/ft}^3} = 14.6 \text{ ft} \quad \text{(Ans)} \\
  p_2 &= \frac{908 \text{ lb/ft}^2}{144 \text{ in.}^2/\text{ft}^2} = 6.31 \text{ lb/in.}^2 
\end{align*}
\]

Observe that if we wish to express these pressures in terms of absolute pressure, we would have to add the local atmospheric pressure (in appropriate units) to the previous results. A further discussion of gage and absolute pressure is given in Section 2.5.

The transmission of pressure through a stationary fluid is the principle upon which many hydraulic devices are based.

2.3.2 Compressible Fluid

We normally think of gases such as air, oxygen, and nitrogen as being compressible fluids since the density of the gas can change significantly with changes in pressure and temperature. Thus, although Eq. 2.4 applies at a point in a gas, it is necessary to consider the possible variation in \( \gamma \) before the equation can be integrated. However, as was discussed in Chapter 1, the specific weights of common gases are small when compared with those of liquids. For example, the specific weight of air at sea level and 60 °F is 0.0763 lb/ft³, whereas the specific weight of water under the same conditions is 62.4 lb/ft³. Since the specific weights of gases are comparatively small, it follows from Eq. 2.4 that the pressure gradient in the vertical direction is correspondingly small, and even over distances of several hundred feet the pressure will remain essentially constant for a gas. This means we can neglect the effect of elevation changes on the pressure in gases in tanks, pipes, and so forth in which the distances involved are small.
For those situations in which the variations in heights are large, on the order of thousands of feet, attention must be given to the variation in the specific weight. As described in Chapter 1, the equation of state for an ideal (or perfect) gas is

\[ p = \rho RT \]

where \( p \) is the absolute pressure, \( R \) is the gas constant, and \( T \) is the absolute temperature. This relationship can be combined with Eq. 2.4 to give

\[ \frac{dp}{dz} = \frac{g\rho}{RT} \]

and by separating variables

\[ \int_{p_1}^{p_2} \frac{dp}{p} = \ln \frac{p_2}{p_1} = -\frac{g}{R} \int_{z_1}^{z_2} \frac{dz}{T} \]

where \( g \) and \( R \) are assumed to be constant over the elevation change from \( z_1 \) to \( z_2 \). Although the acceleration of gravity, \( g \), does vary with elevation, the variation is very small (see Tables C.1 and C.2 in Appendix C), and \( g \) is usually assumed constant at some average value for the range of elevation involved.

Before completing the integration, one must specify the nature of the variation of temperature with elevation. For example, if we assume that the temperature has a constant value \( T_0 \) over the range \( z_1 \) to \( z_2 \) (isothermal conditions), it then follows from Eq. 2.9 that

\[ p_2 = p_1 e^{-\frac{g(z_2 - z_1)}{RT_0}} \]

This equation provides the desired pressure-elevation relationship for an isothermal layer. For nonisothermal conditions a similar procedure can be followed if the temperature-elevation relationship is known, as is discussed in the following section.

**Example 2.2**

The Empire State Building in New York City, one of the tallest buildings in the world, rises to a height of approximately 1250 ft. Estimate the ratio of the pressure at the top of the building to the pressure at its base, assuming the air to be at a common temperature of 59 °F. Compare this result with that obtained by assuming the air to be incompressible with \( \gamma = 0.0765 \) lb/ft³ at 14.7 psi(abs) (values for air at standard conditions).

**Solution**

For the assumed isothermal conditions, and treating air as a compressible fluid, Eq. 2.10 can be applied to yield

\[ \frac{p_2}{p_1} = e^{-\frac{g(z_2 - z_1)}{RT_0}} = \exp \left\{ -\frac{32.2 \text{ ft/s}^2 \times 1250 \text{ ft}}{(1716 \text{ ft-lb/slug} \cdot \text{R})(59 + 460) \text{°R}} \right\} = 0.956 \quad (\text{Ans}) \]

If the air is treated as an incompressible fluid we can apply Eq. 2.5. In this case

\[ p_2 = p_1 - \gamma(z_2 - z_1) \]
or

\[
P_2 = 1 - \frac{\gamma(z_2 - z_1)}{p_1}
\]

\[
= 1 - \frac{(0.0765 \text{ lb/ft}^2)(1250 \text{ ft})}{(14.7 \text{ lb/in.}^2)(144 \text{ in.}^2/\text{ft}^2)} = 0.955
\]

Note that there is little difference between the two results. Since the pressure difference between the bottom and top of the building is small, it follows that the variation in fluid density is small and, therefore, the compressible fluid and incompressible fluid analyses yield essentially the same result.

We see that for both calculations the pressure decreases by less than 5% as we go from ground level to the top of this tall building. It does not require a very large pressure difference to support a 1250-ft-tall column of fluid as light as air. This result supports the earlier statement that the changes in pressures in air and other gases due to elevation changes are very small, even for distances of hundreds of feet. Thus, the pressure differences between the top and bottom of a horizontal pipe carrying a gas, or in a gas storage tank, are negligible since the distances involved are very small.

2.4 Standard Atmosphere

An important application of Eq. 2.9 relates to the variation in pressure in the earth's atmosphere. Ideally, we would like to have measurements of pressure versus altitude over the specific range for the specific conditions (temperature, reference pressure) for which the pressure is to be determined. However, this type of information is usually not available. Thus, a "standard atmosphere" has been determined that can be used in the design of aircraft, missiles, and spacecraft, and in comparing their performance under standard conditions. The concept of a standard atmosphere was first developed in the 1920s, and since that time many national and international committees and organizations have pursued the development of such a standard. The currently accepted standard atmosphere is based on a report published in 1962 and updated in 1976 (see Refs. 1 and 2), defining the so-called U.S. standard atmosphere, which is an idealized representation of middle-latitude, year-round mean conditions of the earth's atmosphere. Several important properties for standard atmospheric conditions at sea level are listed in Table 2.1, and Fig. 2.6 shows the temperature profile for the U.S. standard atmosphere. As is shown in this figure the temperature decreases with altitude in the

<table>
<thead>
<tr>
<th>TABLE 2.1</th>
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</thead>
<tbody>
<tr>
<td>Properties of U.S. Standard Atmosphere at Sea Levela</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Property</th>
<th>SI Units</th>
<th>BG Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature, ( T )</td>
<td>288.15 K (15 °C)</td>
<td>518.67 °R (59.00 °F)</td>
</tr>
<tr>
<td>Pressure, ( p )</td>
<td>101.33 kPa (abs)</td>
<td>2116.2 lb/ft² (abs)</td>
</tr>
<tr>
<td>Density, ( \rho )</td>
<td>1.225 kg/m³</td>
<td>0.002377 slugs/ft³</td>
</tr>
<tr>
<td>Specific weight, ( \gamma )</td>
<td>12.014 N/m³</td>
<td>0.07647 lb/ft³</td>
</tr>
<tr>
<td>Viscosity, ( \mu )</td>
<td>1.789 \times 10^{-5} N-s/m²</td>
<td>3.737 \times 10^{-7} lb-s/ft²</td>
</tr>
</tbody>
</table>

*aAcceleration of gravity at sea level = 9.807 m/s² = 32.174 ft/s².*
region nearest the earth’s surface (troposphere), then becomes essentially constant in the next layer (stratosphere), and subsequently starts to increase in the next layer. Since the temperature variation is represented by a series of linear segments, it is possible to integrate Eq. 2.9 to obtain the corresponding pressure variation. For example, in the troposphere, which extends to an altitude of about 11 km (~36,000 ft), the temperature variation is of the form

\[ T = T_a - \beta z \] (2.11)

where \( T_a \) is the temperature at sea level \( (z = 0) \) and \( \beta \) is the lapse rate (the rate of change of temperature with elevation). For the standard atmosphere in the troposphere, \( \beta = 0.00650 \) K/m or 0.00357 °R/ft.

Equation 2.11 used with Eq. 2.9 yields

\[ p = p_a \left(1 - \frac{\beta z}{T_a} \right)^{\gamma/\beta} \] (2.12)

where \( p_a \) is the absolute pressure at \( z = 0 \). With \( p_a \), \( T_a \), and \( g \) obtained from Table 2.1, and with the gas constant \( R = 286.9 \) J/kg·K or 1716 ft·lb/slug·°R, the pressure variation throughout the troposphere can be determined from Eq. 2.12. This calculation shows that at the outer edge of the troposphere, where the temperature is \( -56.5 \) °C, the absolute pressure is about 23 kPa (3.3 psia). It is to be noted that modern jetliners cruise at approximately this altitude. Pressures at other altitudes are shown in Fig. 2.6, and tabulated values for temperature, acceleration of gravity, pressure, density, and viscosity for the U.S. standard atmosphere are given in Tables C.1 and C.2 in Appendix C.

2.5 Measurement of Pressure

Since pressure is a very important characteristic of a fluid field, it is not surprising that numerous devices and techniques are used in its measurement. As is noted briefly in Chapter 1, the pressure at a point within a fluid mass will be designated as either an absolute pressure or a gage pressure. Absolute pressure is measured relative to a perfect vacuum (absolute zero...
pressure), whereas gage pressure is measured relative to the local atmospheric pressure. Thus, a gage pressure of zero corresponds to a pressure that is equal to the local atmospheric pressure. Absolute pressures are always positive, but gage pressures can be either positive or negative depending on whether the pressure is above atmospheric pressure (a positive value) or below atmospheric pressure (a negative value). A negative gage pressure is also referred to as a suction or vacuum pressure. For example, 10 psi (abs) could be expressed as -4.7 psi (gage), if the local atmospheric pressure is 14.7 psi, or alternatively 4.7 psi suction or 4.7 psi vacuum. The concept of gage and absolute pressure is illustrated graphically in Fig. 2.7 for two typical pressures located at points 1 and 2.

In addition to the reference used for the pressure measurement, the units used to express the value are obviously of importance. As is described in Section 1.5, pressure is a force per unit area, and the units in the BG system are lb/ft$^2$ or lb/in.$^2$, commonly abbreviated psf or psi, respectively. In the SI system the units are N/m$^2$; this combination is called the pascal and written as Pa (1 N/m$^2$ = 1 Pa). As noted earlier, pressure can also be expressed as the height of a column of liquid. Then, the units will refer to the height of the column (in., ft, mm, m, etc.), and in addition, the liquid in the column must be specified (H$_2$O, Hg, etc.). For example, standard atmospheric pressure can be expressed as 760 mm Hg (abs). In this text, pressures will be assumed to be gage pressures unless specifically designated absolute. For example, 10 psi or 100 kPa would be gage pressures, whereas 10 psia or 100 kPa (abs) would refer to absolute pressures. It is to be noted that pressure differences are independent of the reference, so that no special notation is required in this case.

The measurement of atmospheric pressure is usually accomplished with a mercury barometer, which in its simplest form consists of a glass tube closed at one end with the open end immersed in a container of mercury as shown in Fig. 2.8. The tube is initially filled with mercury (inverted with its open end up) and then turned upside down (open end down) with the open end in the container of mercury. The column of mercury will come to an equilibrium position where its weight plus the force due to the vapor pressure (which develops in the space above the column) balances the force due to the atmospheric pressure. Thus,

$$P_{\text{atm}} = \gamma h + P_{\text{vapor}}$$

(2.13)

where $\gamma$ is the specific weight of mercury. For most practical purposes the contribution of the vapor pressure can be neglected since it is very small [for mercury, $P_{\text{vapor}} = 0.000023$ lb/in.$^2$ (abs) at a temperature of 68 °F] so that $P_{\text{atm}} \approx \gamma h$. It is conventional to specify atmospheric pressure in terms of the height, $h$, in millimeters or inches of mercury. Note that if water were used instead of mercury, the height of the column would have to be approximately 34 ft rather than 29.9 in. of mercury for an atmospheric pressure of 14.7 psia! The concept of the mercury barometer is an old one, with the invention of this device attributed to Evangelista Torricelli in about 1644.
EXAMPLE 2.3
A mountain lake has an average temperature of 10 °C and a maximum depth of 40 m. For a barometric pressure of 598 mm Hg, determine the absolute pressure (in pascals) at the deepest part of the lake.

SOLUTION
The pressure in the lake at any depth, h, is given by the equation

$$p = \gamma h + p_0$$

where $p_0$ is the pressure at the surface. Since we want the absolute pressure, $p_0$ will be the local barometric pressure expressed in a consistent system of units; that is

$$p_{\text{barometric}} = 598 \text{ mm} = 0.598 \text{ m}$$

and for $\gamma_{\text{Hg}} = 133 \text{ kN/m}^3$

$$p_0 = (0.598 \text{ m})(133 \text{ kN/m}^3) = 79.5 \text{ kN/m}^2$$

From Table B.2, $\gamma_{\text{H}_2\text{O}} = 9.804 \text{ kN/m}^3$ at 10 °C and therefore

$$p = (9.804 \text{ kN/m}^3)(40 \text{ m}) + 79.5 \text{ kN/m}^2$$

$$= 392 \text{ kN/m}^2 + 79.5 \text{ kN/m}^2 = 472 \text{ kPa (abs)} \quad (\text{Ans})$$

This simple example illustrates the need for close attention to the units used in the calculation of pressure; that is, be sure to use a consistent unit system, and be careful not to add a pressure head (m) to a pressure (Pa).

2.6 Manometry

A standard technique for measuring pressure involves the use of liquid columns in vertical or inclined tubes. Pressure measuring devices based on this technique are called manometers. The mercury barometer is an example of one type of manometer, but there are many other configurations possible, depending on the particular application. Three common types of manometers include the piezometer tube, the U-tube manometer, and the inclined-tube manometer.
To determine pressure from a manometer, simply use the fact that the pressure in the liquid columns will vary hydrostatically.

2.6.1 Piezometer Tube

The simplest type of manometer consists of a vertical tube, open at the top, and attached to the container in which the pressure is desired, as illustrated in Fig. 2.9. Since manometers involve columns of fluids at rest, the fundamental equation describing their use is Eq. 2.8

\[ p = \gamma h + p_0 \]

which gives the pressure at any elevation within a homogeneous fluid in terms of a reference pressure \( p_0 \) and the vertical distance \( h \) between \( p \) and \( p_0 \). Remember that in a fluid at rest pressure will increase as we move downward and will decrease as we move upward. Application of this equation to the piezometer tube of Fig. 2.9 indicates that the pressure \( P_A \) can be determined by a measurement of \( h_1 \) through the relationship

\[ P_A = \gamma_1 h_1 \]

where \( \gamma_1 \) is the specific weight of the liquid in the container. Note that since the tube is open at the top, the pressure \( p_0 \) can be set equal to zero (we are now using gage pressure), with the height \( h_1 \) measured from the meniscus at the upper surface to point (1). Since point (1) and point \( A \) within the container are at the same elevation, \( P_A = P_1 \).

Although the piezometer tube is a very simple and accurate pressure measuring device, it has several disadvantages. It is only suitable if the pressure in the container is greater than atmospheric pressure (otherwise air would be sucked into the system), and the pressure to be measured must be relatively small so the required height of the column is reasonable. Also, the fluid in the container in which the pressure is to be measured must be a liquid rather than a gas.

2.6.2 U-Tube Manometer

To overcome the difficulties noted previously, another type of manometer which is widely used consists of a tube formed into the shape of a U as is shown in Fig. 2.10. The fluid in the manometer is called the gage fluid. To find the pressure \( P_A \) in terms of the various column heights, we start at one end of the system and work our way around to the other end, simply utilizing Eq. 2.8. Thus, for the U-tube manometer shown in Fig. 2.10, we will start at point \( A \) and work around to the open end. The pressure at points \( A \) and (1) are the same, and as we move from point (1) to (2) the pressure will increase by \( \gamma_1 h_1 \). The pressure at point (2) is equal to the pressure at point (3), since the pressures at equal elevations in a continuous mass of fluid at rest must be the same. Note that we could not simply "jump across" from point (1) to a point at the same elevation in the right-hand tube since these would not be points within the same continuous mass of fluid. With the pressure at point (3) specified we now move to the open end where the pressure is zero. As we move vertically upward the pressure decreases by an amount \( \gamma_2 h_2 \). In equation form these various steps can be expressed as

\[ P_A + \gamma_1 h_1 - \gamma_2 h_2 = 0 \]
and, therefore, the pressure $p_A$ can be written in terms of the column heights as

$$p_A = \gamma_2 h_2 - \gamma_1 h_1$$

(2.14)

A major advantage of the U-tube manometer lies in the fact that the gage fluid can be different from the fluid in the container in which the pressure is to be determined. For example, the fluid in $A$ in Fig. 2.10 can be either a liquid or a gas. If $A$ does contain a gas, the contribution of the gas column, $\gamma_1 h_1$, is almost always negligible so that $p_A = p_2$ and in this instance Eq. 2.14 becomes

$$p_A = \gamma_2 h_2$$

Thus, for a given pressure the height, $h_2$, is governed by the specific weight, $\gamma_2$, of the gage fluid used in the manometer. If the pressure $p_A$ is large, then a heavy gage fluid, such as mercury, can be used and a reasonable column height (not too long) can still be maintained. Alternatively, if the pressure $p_A$ is small, a lighter gage fluid, such as water, can be used so that a relatively large column height (which is easily read) can be achieved.

**Example 2.4**

A closed tank contains compressed air and oil ($SG_{oil} = 0.90$) as is shown in Fig. E2.4. A U-tube manometer using mercury ($SG_{Hg} = 13.6$) is connected to the tank as shown. For column heights $h_1 = 36$ in., $h_2 = 6$ in., and $h_3 = 9$ in., determine the pressure reading (in psi) of the gage.
SOLUTION

Following the general procedure of starting at one end of the manometer system and working around to the other, we will start at the air–oil interface in the tank and proceed to the open end where the pressure is zero. The pressure at level (1) is

\[ p_1 = P_{\text{air}} + \gamma_{\text{oil}}(h_1 + h_2) \]

This pressure is equal to the pressure at level (2), since these two points are at the same elevation in a homogeneous fluid at rest. As we move from level (2) to the open end, the pressure must decrease by \( \gamma_{\text{Hg}}h_3 \), and at the open end the pressure is zero. Thus, the manometer equation can be expressed as

\[ P_{\text{air}} + \gamma_{\text{oil}}(h_1 + h_2) - \gamma_{\text{Hg}}h_3 = 0 \]

or

\[ P_{\text{air}} + (S_{\text{G_oil}})(\gamma_{\text{HgO}})(h_1 + h_2) - (S_{\text{G_Hg}})(\gamma_{\text{HgO}})h_3 = 0 \]

For the values given

\[ P_{\text{air}} = -(0.9)(62.4 \text{ lb/ft}^3) \left( \frac{36 + 6}{12} \text{ ft} \right) + (13.6)(62.4 \text{ lb/ft}^3) \left( \frac{9}{12} \text{ ft} \right) \]

so that

\[ P_{\text{air}} = 440 \text{ lb/ft}^2 \]

Since the specific weight of the air above the oil is much smaller than the specific weight of the oil, the gage should read the pressure we have calculated; that is,

\[ P_{\text{gage}} = \frac{440 \text{ lb/ft}^2}{144 \text{ in.}^2/\text{ft}^2} = 3.06 \text{ psi} \]  \( \text{(Ans)} \)

The U-tube manometer is also widely used to measure the difference in pressure between two containers or two points in a given system. Consider a manometer connected between containers \( A \) and \( B \) as is shown in Fig. 2.11. The difference in pressure between \( A \) and \( B \) can be found by again starting at one end of the system and working around to the other end. For example, at \( A \) the pressure is \( P_A \), which is equal to \( P_1 \), and as we move to point (2) the pressure increases by \( \gamma_1h_1 \). The pressure at \( P_2 \) is equal to \( P_3 \), and as we move upward...
to point (4) the pressure decreases by $\gamma_2 h_2$. Similarly, as we continue to move upward from point (4) to (5) the pressure decreases by $\gamma_3 h_3$. Finally, $p_5 = p_B$, since they are at equal elevations. Thus,

$$p_A + \gamma_1 h_1 - \gamma_2 h_2 - \gamma_3 h_3 = p_B$$

and the pressure difference is

$$p_A - p_B = \gamma_2 h_2 + \gamma_3 h_3 - \gamma_1 h_1$$

When the time comes to substitute in numbers, be sure to use a consistent system of units!

Capillarity due to surface tension at the various fluid interfaces in the manometer is usually not considered, since for a simple U-tube with a meniscus in each leg, the capillary effects cancel (assuming the surface tensions and tube diameters are the same at each meniscus), or we can make the capillary rise negligible by using relatively large bore tubes (with diameters of about 0.5 in. or larger). Two common gage fluids are water and mercury. Both give a well-defined meniscus (a very important characteristic for a gage fluid) and have well-known properties. Of course, the gage fluid must be immiscible with respect to the other fluids in contact with it. For highly accurate measurements, special attention should be given to temperature since the various specific weights of the fluids in the manometer will vary with temperature.

**Example 2.5**

As will be discussed in Chapter 3, the volume rate of flow, $Q$, through a pipe can be determined by means of a flow nozzle located in the pipe as illustrated in Fig. E2.5. The nozzle creates a pressure drop, $p_A - p_B$, along the pipe which is related to the flow through the equation $Q = K \sqrt{p_A - p_B}$, where $K$ is a constant depending on the pipe and nozzle size. The pressure drop is frequently measured with a differential U-tube manometer of the type illustrated. (a) Determine an equation for $p_A - p_B$ in terms of the specific weight of the flowing fluid, $\gamma_1$, the specific weight of the gage fluid, $\gamma_2$, and the various heights indicated. (b) For $\gamma_1 = 9.80 \text{ kN}/\text{m}^3$, $\gamma_2 = 15.6 \text{ kN}/\text{m}^3$, $h_1 = 1.0 \text{ m}$, and $h_2 = 0.5 \text{ m}$, what is the value of the pressure drop, $p_A - p_B$?

**Solution**

(a) Although the fluid in the pipe is moving, the fluids in the columns of the manometer are at rest so that the pressure variation in the manometer tubes is hydrostatic. If we start at point A and move vertically upward to level (1), the pressure will decrease by $\gamma_1 h_1$ and will be equal to the pressure at (2) and at (3). We can now move from (3) to (4) where the pressure has been further reduced by $\gamma_2 h_2$. The pressures at levels (4) and (5) are equal, and as we move from (5) to B the pressure will increase by $\gamma_1 (h_1 + h_2)$. 

![Flow nozzle](image-url)
Thus, in equation form
\[ p_A - \gamma_1 h_1 - \gamma_2 h_2 + \gamma_1 (h_1 + h_2) = p_B \]
or
\[ p_A - p_B = h_2 (\gamma_2 - \gamma_1) \]  
(Ans)

It is to be noted that the only column height of importance is the differential reading, \( h_2 \). The differential manometer could be placed 0.5 or 5.0 m above the pipe \( (h_1 = 0.5 \text{ m or } h_1 = 5.0 \text{ m}) \) and the value of \( h_2 \) would remain the same. Relatively large values for the differential reading \( h_2 \) can be obtained for small pressure differences, \( p_A - p_B \), if the difference between \( \gamma_1 \) and \( \gamma_2 \) is small.

(b) The specific value of the pressure drop for the data given is
\[ p_A - p_B = (0.5 \text{ m})(15.6 \text{ kN/m}^3 - 9.80 \text{ kN/m}^3) \]
\[ = 2.90 \text{ kPa} \]  
(Ans)

### 2.6.3 Inclined-Tube Manometer

To measure small pressure changes, a manometer of the type shown in Fig. 2.12 is frequently used. One leg of the manometer is inclined at an angle \( \theta \), and the differential reading \( \ell_2 \) is measured along the inclined tube. The difference in pressure \( p_A - p_B \) can be expressed as
\[ p_A + \gamma_1 h_1 - \gamma_2 \ell_2 \sin \theta - \gamma_3 h_3 = p_B \]
or
\[ p_A - p_B = \gamma_2 \ell_2 \sin \theta + \gamma_3 h_3 - \gamma_1 h_1 \]  
(2.15)

where it is to be noted the pressure difference between points (1) and (2) is due to the vertical distance between the points, which can be expressed as \( \ell_2 \sin \theta \). Thus, for relatively small angles the differential reading along the inclined tube can be made large even for small pressure differences. The inclined-tube manometer is often used to measure small differences in gas pressures so that if pipes \( A \) and \( B \) contain a gas then
\[ p_A - p_B = \gamma_2 \ell_2 \sin \theta \]
or
\[ \ell_2 = \frac{p_A - p_B}{\gamma_2 \sin \theta} \]  
(2.16)
where the contributions of the gas columns \( h_1 \) and \( h_3 \) have been neglected. Equation 2.16 shows that the differential reading \( \ell_2 \) (for a given pressure difference) of the inclined-tube manometer can be increased over that obtained with a conventional U-tube manometer by the factor \( 1/\sin \theta \). Recall that \( \sin \theta \to 0 \) as \( \theta \to 0 \).

2.7 Mechanical and Electronic Pressure Measuring Devices

Although manometers are widely used, they are not well suited for measuring very high pressures, or pressures that are changing rapidly with time. In addition, they require the measurement of one or more column heights, which, although not particularly difficult, can be time consuming. To overcome some of these problems numerous other types of pressure-measuring instruments have been developed. Most of these make use of the idea that when a pressure acts on an elastic structure the structure will deform, and this deformation can be related to the magnitude of the pressure. Probably the most familiar device of this kind is the Bourdon pressure gage, which is shown in Fig. 2.13a. The essential mechanical element in this gage is the hollow, elastic curved tube (Bourdon tube) which is connected to the pressure source as shown in Fig. 2.13b. As the pressure within the tube increases the tube tends to straighten, and although the deformation is small, it can be translated into the motion of a pointer on a dial as illustrated. Since it is the difference in pressure between the outside of the tube (atmospheric pressure) and the inside of the tube that causes the movement of the tube, the indicated pressure is gage pressure. The Bourdon gage must be calibrated so that the dial reading can directly indicate the pressure in suitable units such as psi, psf, or pascals. A zero reading on the gage indicates that the measured pressure is equal to the local atmospheric pressure. This type of gage can be used to measure a negative gage pressure (vacuum) as well as positive pressures.

The aneroid barometer is another type of mechanical gage that is used for measuring atmospheric pressure. Since atmospheric pressure is specified as an absolute pressure, the conventional Bourdon gage is not suitable for this measurement. The common aneroid barometer contains a hollow, closed, elastic element which is evacuated so that the pressure inside the element is near absolute zero. As the external atmospheric pressure changes, the element deflects, and this motion can be translated into the movement of an attached dial. As with the Bourdon gage, the dial can be calibrated to give atmospheric pressure directly, with the usual units being millimeters or inches of mercury.
For many applications in which pressure measurements are required, the pressure must be measured with a device that converts the pressure into an electrical output. For example, it may be desirable to continuously monitor a pressure that is changing with time. This type of pressure measuring device is called a pressure transducer, and many different designs are used. One possible type of transducer is one in which a Bourdon tube is connected to a linear variable differential transformer (LVDT), as is illustrated in Fig. 2.14. The core of the LVDT is connected to the free end of the Bourdon so that as a pressure is applied the resulting motion of the end of the tube moves the core through the coil and an output voltage develops. This voltage is a linear function of the pressure and could be recorded on an oscillograph or digitized for storage or processing on a computer.

One disadvantage of a pressure transducer using a Bourdon tube as the elastic sensing element is that it is limited to the measurement of pressures that are static or only changing slowly (quasistatic). Because of the relatively large mass of the Bourdon tube, it cannot respond to rapid changes in pressure. To overcome this difficulty a different type of transducer is used in which the sensing element is a thin, elastic diaphragm which is in contact with the fluid. As the pressure changes, the diaphragm deflects, and this deflection can be sensed and converted into an electrical voltage. One way to accomplish this is to locate strain gages either on the surface of the diaphragm not in contact with the fluid, or on an element attached to the diaphragm. These gages can accurately sense the small strains induced in the diaphragm and provide an output voltage proportional to pressure. This type of transducer is capable of measuring accurately both small and large pressures, as well as both static and dynamic pressures. For example, strain-gage pressure transducers of the type shown in Fig. 2.15 are used to measure arterial blood pressure, which is a relatively small pressure that varies periodically with a fundamental frequency of about 1 Hz. The transducer is usually connected to the blood vessel by means of a liquid-filled, small diameter tube called a pressure catheter.

Although the strain-gage type of transducers can be designed to have very good frequency response (up to approximately 10 kHz), they become less sensitive at the higher frequencies since the diaphragm must be made stiffer to achieve the higher frequency response. As an alternative the diaphragm can be constructed of a piezoelectric crystal to be used as both the elastic element and the sensor. When a pressure is applied to the crystal a voltage develops because of the deformation of the crystal. This voltage is directly related to the applied pressure. Depending on the design, this type of transducer can be used to measure both very low and high pressures (up to approximately 100,000 psi) at high frequencies. Additional information on pressure transducers can be found in Refs. 3, 4, and 5.
When a surface is submerged in a fluid, forces develop on the surface due to the fluid. The determination of these forces is important in the design of storage tanks, ships, dams, and other hydraulic structures. For fluids at rest we know that the force must be perpendicular to the surface since there are no shearing stresses present. We also know that the pressure will vary linearly with depth if the fluid is incompressible. For a horizontal surface, such as the bottom of a liquid-filled tank (Fig. 2.16), the magnitude of the resultant force is simply \( F_R = pA \), where \( p \) is the uniform pressure on the bottom and \( A \) is the area of the bottom. For the open tank shown, \( p = \gamma h \). Note that if atmospheric pressure acts on both sides of the bottom, as is illustrated, the resultant force on the bottom is simply due to the liquid in the tank. Since the pressure is constant and uniformly distributed over the bottom, the resultant force acts through the centroid of the area as shown in Fig. 2.16.

For the more general case in which a submerged plane surface is inclined, as is illustrated in Fig. 2.17, the determination of the resultant force acting on the surface is more involved. For the present we will assume that the fluid surface is open to the atmosphere. Let the plane in which the surface lies intersect the free surface at \( 0 \) and make an angle \( \theta \) with the horizontal.